Solitons, kinks and extended hadron model based on the generalized sine-Gordon theory

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Abstract: The solitons and kinks of the generalized $sl(3,\mathbb{C})$ sine-Gordon (GSG) model are explicitly obtained through the hybrid of the Hirota and dressing methods in which the *tau* functions play an important role. The various properties are investigated, such as the potential vacuum structure, the soliton and kink solutions, and the soliton masses formulae. As a reduced submodel we obtain the double sine-Gordon model. Moreover, we provide the algebraic construction of the $sl(3,\mathbb{C})$ affine Toda model coupled to matter (Dirac spinor) (ATM) and through a gauge fixing procedure we obtain the classical version of the generalized $sl(3,\mathbb{C})$ sine-Gordon model (cGSG) which completely decouples from the Dirac spinors. In the spinor sector we are left with Dirac fields coupled to cGSG fields. Based on the equivalence between the U(1) vector and topological currents it is shown the confinement of the spinors inside the solitons and kinks of the cGSG model providing an extended hadron model for "quark" confinement.

Keywords: Nonperturbative Effects, Integrable Field Theories, Confinement, Field Theories in Lower Dimensions.

Contents

1. Introduction

The sine-Gordon model (SG) has been studied over the decades due to its many properties and mathematical structures such as integrability and soliton solutions. It can be used as a toy model for non-perturbative quantum field theory phenomena. In this context, some extensions and modifications of the SG model deserve attention. An extension taking multi-frequency terms as the potential has been investigated in connection to various physical applications [1 – 4].

On the other hand, an extension defined for multi-fields is the so-called generalized sine-Gordon model (GSG) which has been found in the study of the strong/weak coupling sectors of the so-called $sl(N, \mathbb{C})$ affine Toda coupled to matter fields (ATM) theory [5, 6]. In connection to these developments, the bosonization process of the multi-flavor massive Thirring model (GMT) provides the quantum version of the (GSG) model [7]. The GSG model provides a framework to obtain (multi-)soliton solutions for unequal mass parameters of the fermions in the GMT sector and study the spectrum and their interactions. The extension of this picture to the NC space-time has been addressed (see [8] and references therein).

Coupled systems of scalar fields have been investigated by many authors [9 – 14]. One of the motivations was the study of topological defects in relativistic field theories; since realistic theories involve more than one scalar field, the multi-field sine-Gordon theories with kink-type exact solutions deserve some attention. The interest in the study of the classical limit of string theory on determined backgrounds has recently been greatly stimulated in connection to integrability. It has been established that the classical string on $R \times S^2$ is essentially equivalent to the sine-Gordon integrable system [15]. More recently, on $R \times S^3$ background utilizing the Pohlmeyers reduction it has been obtained a family of classical string solutions called dyonic giant magnons which were associated with solitons of complex sine-Gordon equations [16]. String theory on $R \times S^{N-1}$ is classically equivalent to the so-called $SO(N)$ symmetric space sine-Gordon model (SSG) [17].

In this paper we study the spectrum of solitons and kinks of the GSG model proposed in [5 – 7] and consider the closely related ATM model from which one gets the classical GSG model (cGSG) through a gauge fixing procedure. Some reductions of the GSG model to one-field theory lead to the usual SG model and to the so-called multi-frequency sine-Gordon models. In particular, the double (two-frequency) sine-Gordon model (DSG) appears in a reduction of the $sl(3,\mathbb{C})$ GSG model. The DSG theory is a nonintegrable quantum field theory with many physical applications [3, 4].

Once a convenient gauge fixing is performed by setting to constant some spinor bilinears in the ATM model we are left with two sectors: the cGSG model which completely decouples from the spinors and a system of Dirac spinors coupled to the cGSG fields. Following the references [18, 19] in which a $1 + 1$-dimensional bag model for quark confinement is considered, we follow their ideas and generalize for multi-flavor Dirac spinors coupled to cGSG solitons and kinks. The first reference considers a model similar to the $sl(2)$ ATM theory, and in the second one the DSG kink is proposed as an extended hadron model.

In the next section we define the $sl(3,\mathbb{C})$ GSG model and study its properties such as the vacuum structure and the soliton, kink and bounce type solutions. In section 3 we consider the $sl(3,\mathbb{C})$ affine Toda model coupled to matter and obtain the cGSG model through a gauge fixing procedure. It is discussed the physical soliton spectrum of the gauge fixed model. In section 4 the topological charges are introduced, as well as the idea of baryons as solitons (or kinks), and the quark confinement mechanism is discussed. The discussion section outlines possible directions for future research, in particular, the GSG application to QCD$_2$. In appendix A we provide the zero curvature formulation of the $sl(3,\mathbb{C})$ ATM model.

2. The model

The generalized sine-Gordon model (GSG) related to $sl(N,\mathbb{C})$ is defined by [5 – 7]

$$S = \int d^2x \sum_{i=1}^{N_f} \left[\frac{1}{2} (\partial_\mu \Phi_i)^2 + \mu_i \left(\cos \beta_i \Phi_i - 1 \right) \right]. \tag{2.1}$$

The Φ_i fields in (2.1) satisfy the constraints

$$\Phi_p = \sum_{i=1}^{N-1} \sigma_{pi} \Phi_i, \quad p = N, N+1, \ldots, N_f, \quad N_f = \frac{N(N-1)}{2}, \tag{2.2}$$

where σ_{pi} are some constant parameters and N_f is the number of positive roots of the Lie algebra $sl(N, \mathbb{C})$. In the context of the Lie algebraic construction of the GSG system these constraints arise from the relationship between the positive and simple roots of $sl(N, \mathbb{C})$. Thus, in (2.1) we have $(N-1)$ independent fields related to the number of simple roots of the $sl(N, \mathbb{C})$ Lie algebra.

We consider the $sl(3, \mathbb{C})$ algebra, since in this case one has two simple roots there are two independent real fields, $\varphi_{1,2}$, associated to them such that

$$\Phi_1 = 2\varphi_1 - \varphi_2; \quad \Phi_2 = 2\varphi_2 - \varphi_1; \quad \Phi_3 = r\,\varphi_1 + s\,\varphi_2, \quad s, r \in \mathbb{R} \tag{2.3}$$

which must satisfy the constraint

$$\beta_3 \Phi_3 = \delta_1 \beta_1 \Phi_1 + \delta_2 \beta_2 \Phi_2, \quad \beta_i \equiv \beta_0 \nu_i, \tag{2.4}$$

where $\beta_0, \nu_i, \delta_1, \delta_2$ are some real numbers. The Φ fields dependence on the φ ones will be explained in the context of the Lie algebraic construction of the classical version of the model in sections 3 and 4. In view of the definitions above, the $sl(3, \mathbb{C})$ GSG model can be regarded as three usual sine-Gordon models coupled through the linear constraint (2.4).

Taking into account (2.3)–(2.4) and the fact that the fields φ_1 and φ_2 are independent we may get the relationships

$$\nu_2 \delta_2 = \rho_0 \nu_1 \delta_1 \quad \nu_3 = \frac{1}{r+s}(\nu_1 \delta_1 + \nu_2 \delta_2); \quad \rho_0 \equiv \frac{2s+r}{2r+s} \tag{2.5}$$

The $sl(3, \mathbb{C})$ model has a potential density

$$V[\varphi_i] = \sum_{i=1}^{3} \mu_i \left(1 - \cos \beta_i \Phi_i\right) \tag{2.6}$$

The GSG model has been found in the process of bosonization of the generalized massive Thirring model (GMT) [7]. The GMT model is a multiflavor extension of the usual massive Thirring model incorporating massive fermions with current-current interactions between them. In the $sl(3, \mathbb{C})$ construction of [7] the parameters δ_i depend on the couplings β_i and they satisfy certain relationship. This is obtained by assuming $\mu_i > 0$ and the zero of the potential given for $\Phi_i = \frac{2\pi}{\beta_i} n_i$, which substituted into (2.4) provides

$$n_1 \delta_1 + n_2 \delta_2 = n_3, \quad n_i \in \mathbb{Z} \tag{2.7}$$

The last relation combined with (2.5) gives

$$(2r+s)\frac{n_1}{\nu_1} + (2s+r)\frac{n_2}{\nu_2} = 3\frac{n_3}{\nu_3}. \tag{2.8}$$

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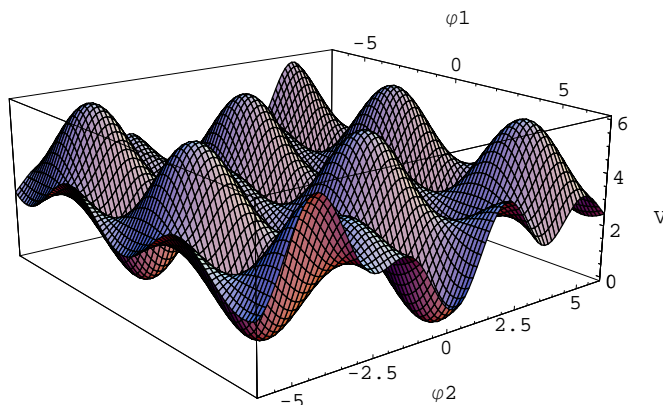
Figure 1: GSG potential V for the parameter values $\nu_1 = 1/2$, $\delta_1 = 2$, $\delta_2 = 1$, $\nu_2 = 1$, $r = s = 1$, $\beta_0 = 1$, $\mu_1 = \mu_2 = 1$.

The periodicity of the potential implies an infinitely degenerate ground state and then the theory supports topologically charged excitations. A typical potential is plotted in figure 1. The vacuum configuration is related to the fundamental weights (see sections 3, 4 and the appendix). For the moment, consider the fields Φ_1 and Φ_2 and the vacuum lattice defined by

$$(\Phi_1,\,\Phi_2) = \frac{2\pi}{\beta_0}\left(\frac{n_1}{\nu_1},\,\frac{n_2}{\nu_2}\right), \quad n_a \in \mathbb{Z}. \tag{2.9}$$

It is convenient to write the equations of motion in terms of the independent fields φ_1 and φ_2

$$\partial^2\varphi_1 = -\mu_1\beta_1\Delta_{11}\sin[\beta_1(2\varphi_1 - \varphi_2)] - \mu_2\beta_2\Delta_{12}\sin[\beta_2(2\varphi_2 - \varphi_1)] +$$
$$\mu_3\beta_3\Delta_{13}\sin[\beta_3(r\varphi_1 + s\varphi_2)] \tag{2.10}$$

$$\partial^2\varphi_2 = -\mu_1\beta_1\Delta_{21}\sin[\beta_1(2\varphi_1 - \varphi_2)] - \mu_2\beta_2\Delta_{22}\sin[\beta_2(2\varphi_2 - \varphi_1)] +$$
$$\mu_3\beta_3\Delta_{23}\sin[\beta_3(r\varphi_1 + s\varphi_2)], \tag{2.11}$$

where

$A = \beta_0^2\nu_1^2(4 + \delta^2 + \delta_1^2\rho_1^2 r^2)$, $B = \beta_0^2\nu_1^2(1 + 4\delta^2 + \delta_1^2\rho_1^2 s^2)$, $C = \beta_0^2\nu_1^2(2 + 2\delta^2 + \delta_1^2\rho_1^2 r\,s)$,

$\Delta_{11} = (C - 2B)/\Delta$, $\Delta_{12} = (B - 2C)/\Delta$, $\Delta_{13} = (r\,B + s\,C)/\Delta$,

$\Delta_{21} = (A - 2C)/\Delta$, $\Delta_{22} = (C - 2A)/\Delta$, $\Delta_{23} = (r\,C + s\,A)/\Delta$

$\Delta = C^2 - AB$, $\delta = \dfrac{\delta_1}{\delta_2}\rho_0$, $\rho_1 = \dfrac{3}{2r + s}$

Notice that the eqs. of motion (2.10)–(2.11) exhibit the symmetries

$$\varphi_1 \leftrightarrow \varphi_2, \quad \mu_1 \leftrightarrow \mu_2, \quad \nu_1 \leftrightarrow \nu_2, \quad \delta_1 \leftrightarrow \delta_2, \quad r \leftrightarrow s; \tag{2.12}$$
$$\text{and} \quad \varphi_a \leftrightarrow -\varphi_a, \quad a = 1, 2 \tag{2.13}$$

Some type of coupled sine-Gordon models have been considered in connection to various interesting physical problems [20]. For example a system of two coupled SG models has

been proposed in order to describe the dynamics of soliton excitations in deoxyribonucleic acid (DNA) double helices [21]. In general these type of equations have been solved by perturbation methods around decoupled sine-Gordon exact solitons.

The system of equations (2.10)–(2.11) for certain choice of the parameters r and s will be derived in section 3 in the context of the $sl(3)$ ATM type models, in which the fields φ_1 and φ_2 couple to some Dirac spinors in such a way that the model exhibits a local gauge invariance. The ATM relevant equations of motion have been solved using a hybrid of the Hirota and Dressing methods [22]. However, in this reference the physical spectrum of solitons and kinks of the theory, related to a convenient gauge fixing of the model, have not been discussed, even though the topological and Noether currents equivalence has been verified. The appearance of the so-called tau functions, in order to find soliton solutions in integrable models, is quite a general result in the both Dressing and Hirota approaches. In this section, we will find soliton and kink type solutions of the GSG model (2.10)–(2.11) and closely follow the spirit of the above hybrid method approach to find soliton solutions.

The general tau function for an $n-$soliton solution of the *gauge unfixed* ATM model has the form [22, 23]

$$\tau = \sum_{p_1,\ldots,p_n=0}^{2} c_{p_1,\ldots,p_n} \exp[p_1 \Gamma_{i_1}(z_1) + \cdots + p_n \Gamma_{i_n}(z_n)], \quad z_i = \gamma_i(x - v_i t), \quad (2.14)$$
$$c_{p_1,\ldots,p_n} \in \mathbb{C}$$

Since the GSG model describes the strong coupling sector (soliton spectrum) of the ATM model [5, 6] then one can guess the following Ansatz for the tau functions of the GSG model

$$e^{-i\beta_0 \frac{\varphi_1}{2}} = \frac{\tau_1}{\tau_0}, \quad e^{-i\beta_0 \frac{\varphi_2}{2}} = \frac{\tau_2}{\tau_0}, \quad (2.15)$$

where the tau functions τ_i $(i = 0, 1, 2)$ are assumed to be of the form (2.14). The relationships (2.15) mimic the ones appearing in section 3 below for the relations between the ATM scalar fields and the relevant three tau functions (see eq. (3.32)). We have two independent fields $\varphi_{1,2}$ in the model (2.10)–(2.11). However, in (2.15) we have introduced three tau-functions. The one equation missing arises from an algebraic relation between the tau functions $\tau_{1,2}$. In fact, from the relationships (2.3), (2.4) and (2.5), together with (2.15) one deduces that the tau functions $\tau_{1,2}$ must be of the form

$$\tau_a = |\tau| e^{i\zeta_a}; \quad (a = 1, 2), \quad \zeta_1 - \zeta_2 = \frac{2\pi n}{k}, \quad (2.16)$$
$$k = 2[(2r + s)\delta_2 \nu_2 - (2s + r)\delta_1 \nu_1], \quad n \in \mathbb{Z}.$$

Assuming that the fields φ_a are real one has that (since $-i\varphi_a = \log|\frac{\tau_a}{\tau_0}| + i\arg\frac{\tau_a}{\tau_0}$)

$$|\tau_0| = |\tau_a| = |\tau|, \quad (a = 1, 2) \quad (2.17)$$

where $|\tau|$ has been defined in (2.16).

Therefore, we are parameterizing two independent real fields $\varphi_{1,2}$ with three tau functions satisfying the constraints (2.16) and (2.17). From (2.15) one can write

$$\varphi_{1,2} = \frac{4}{\beta_0}\arctan\left[\frac{e_1\left([Re(\tau_{1,2})]^2+[Im(\tau_{1,2})]^2\right)-(Re(\tau_{1,2})Re(\tau_0)+Im(\tau_{1,2})Im(\tau_0))}{\left[Im(\tau_{1,2})Re(\tau_0)-Re(\tau_{1,2})Im(\tau_0)\right]}\right],$$
(2.18)

$$e_1 = \pm 1, \quad \text{or}$$

$$\varphi_a = \zeta_0 - \zeta_a + n_a\pi, \; n_a \in \mathbb{Z}, \quad a = 1, 2;$$
(2.19)

where the phases $\zeta_{1,2}$ were defined in (2.16), and ζ_0 corresponds to the definition $\tau_0 = |\tau_0|e^{i\zeta_0}$.

According to the symmetry (2.13) of the system of eqs. (2.10)–(2.11) there will be another solution written as

$$e^{-i\beta_0\frac{\varphi_1'}{2}} = \frac{\tau_0}{\tau_1}, \quad e^{-i\beta_0\frac{\varphi_2'}{2}} = \frac{\tau_0}{\tau_2},$$
(2.20)

for the same tau functions $\tau_i \; (i = 0, 1, 2)$.

We will see that the Ansatz (2.15) provides soliton and kink type solutions of the model (2.10)–(2.11), in this way justifying *a posteriori* the assumption made for the number and form of the tau functions.

According to the Hirota method one substitutes the relations between the fields and tau functions (2.15) into the equations of motion (2.10)–(2.11), so in terms of the tau functions these equations become

$$\frac{2i}{\beta_0^2}\left[\frac{\partial^2\tau_1}{\tau_1} - \frac{(\partial\tau_1)^2}{\tau_1^2} - \frac{\partial^2\tau_0}{\tau_0} + \frac{(\partial\tau_0)^2}{\tau_0^2}\right] + \frac{\beta_1\mu_1\Delta_{11}}{2i}\left[\frac{(\tau_2\tau_0)^{4\nu_1} - \tau_1^{8\nu_1}}{(\tau_2\tau_0)^{2\nu_1}\tau_1^{4\nu_1}}\right] +$$
$$\frac{\beta_2\mu_2\Delta_{12}}{2i}\left[\frac{(\tau_1\tau_0)^{4\nu_2} - \tau_2^{8\nu_2}}{(\tau_1\tau_0)^{2\nu_2}\tau_2^{4\nu_2}}\right] - \frac{\beta_3\mu_3\Delta_{13}}{2i}\left[\frac{(\tau_0)^{4\nu_3(r+s)} - \tau_1^{4r\nu_3}\tau_2^{4s\nu_3}}{(\tau_2)^{2s\nu_3}(\tau_1)^{2r\nu_3}\tau_0^{2\nu_3(r+s)}}\right] = 0, \quad (2.21)$$

$$\frac{2i}{\beta_0^2}\left[\frac{\partial^2\tau_2}{\tau_2} - \frac{(\partial\tau_2)^2}{\tau_2^2} - \frac{\partial^2\tau_0}{\tau_0} + \frac{(\partial\tau_0)^2}{\tau_0^2}\right] + \frac{\beta_1\mu_1\Delta_{21}}{2i}\left[\frac{(\tau_2\tau_0)^{4\nu_1} - \tau_1^{8\nu_1}}{(\tau_2\tau_0)^{2\nu_1}\tau_1^{4\nu_1}}\right] +$$
$$\frac{\beta_2\mu_2\Delta_{22}}{2i}\left[\frac{(\tau_1\tau_0)^{4\nu_2} - \tau_2^{8\nu_2}}{(\tau_1\tau_0)^{2\nu_2}\tau_2^{4\nu_2}}\right] - \frac{\beta_3\mu_3\Delta_{23}}{2i}\left[\frac{(\tau_0)^{4\nu_3(r+s)} - \tau_1^{4r\nu_3}\tau_2^{4s\nu_3}}{(\tau_2)^{2s\nu_3}(\tau_1)^{2r\nu_3}\tau_0^{2\nu_3(r+s)}}\right] = 0. \quad (2.22)$$

These equations will be used to implement a computer program for algebraic manipulations like MAPLE in order to verify the soliton type solutions provided the relevant tau functions are supplied. We will see that the 1-soliton and 1-kink type solutions are related to half-integer or integer values of the parameters ν_i and the values $r, s = 0, 1$. In the next subsections we will write the 1-soliton(antisoliton), 1-kink(antikink) and bounce type solutions.

2.1 One soliton/antisoliton pair associated to φ_1

Consider the tau functions

$$\tau_0 = 1 + i\,d\exp[\gamma(x-vt)]; \quad \tau_1 = 1 - i\,d\exp[\gamma(x-vt)];$$
(2.23)

$$\tau_2 = 1 + i\,d\exp[\gamma(x-vt)].$$
(2.24)

This choice satisfies the system of equations (2.21)–(2.22) for the set of parameters

$$\nu_1 = 1/2, \ \delta_1 = 2, \ \delta_2 = 1, \ \nu_2 = 1, \ \nu_3 = 1, \ r = 1. \tag{2.25}$$

provided that

$$13\mu_3 = 5\mu_2 - 4\mu_1, \ \ \gamma_1^2 = \frac{1}{13}(6\mu_2 + 3\mu_1). \tag{2.26}$$

Now, taking $e_1 = 1$ in eq. (2.19) and the relation (2.18) one has

$$\varphi_1 = -\frac{4}{\beta_0}\arctan\{d\exp[\gamma_1(x - vt)]\}, \ \ \varphi_2 = 0. \tag{2.27}$$

This solution is precisely the sine-Gordon 1-antisoliton associated to the field φ_1 with mass $M_1 = \frac{8\gamma_1}{\beta_0^2}$. This solution corresponds to the relations (2.20). We plot an antisoliton of this type in figure 3.

2.2 One soliton/antisoliton pair associated to φ_2

Next, let us consider the tau functions

$$\tau_0 = 1 + i\,d\exp[\gamma(x - vt)], \ \ \tau_1 = 1 + i\,d\exp[\gamma(x - vt)], \tag{2.28}$$
$$\tau_2 = 1 - i\,d\exp[\gamma(x - vt)] \tag{2.29}$$

This set of tau functions solves the system (2.21)–(2.22) for the choice of parameters

$$\nu_1 = 1, \ \delta_1 = 1, \ \delta_2 = 2, \ \nu_2 = 1/2, \ \nu_3 = 1, \ s = 1 \tag{2.30}$$

provided that

$$13\mu_3 = 5\mu_1 - 4\mu_2, \ \ \gamma_2^2 = \frac{1}{13}(6\mu_1 + 3\mu_2) \tag{2.31}$$

Now, choose $e_1 = 1$ in (2.19) and through (2.18) one can get

$$\varphi_2 = -\frac{4}{\beta_0}\arctan\{d\exp[\gamma_2(x - vt)]\}, \ \ \varphi_1 = 0 \tag{2.32}$$

Similarly, this is the sine-Gordon 1-antisoliton associated to the field φ_2 with mass $M_2 = \frac{8\gamma_2}{\beta_0^2}$ and its profile is of the type shown in Fig 3. Likewise, this solution corresponds to the relations (2.20).

2.3 Two 1-soliton/1-antisoliton pairs associated to $\hat{\varphi}_A \equiv \varphi_1 = \varphi_2$ $(A = 1, 2)$

Now, let us consider the tau functions

$$\tau_0 = 1 + i\,d\,\exp[\gamma(x - vt)], \ \ \tau_1 = 1 - i\,d\,\exp[\gamma(x - vt)], \tag{2.33}$$
$$\tau_2 = 1 - i\,d\,\exp[\gamma(x - vt)]. \tag{2.34}$$

This choice satisfies (2.21)–(2.22) for

$$\nu_1 = 1, \ \delta_1 = 1/2, \ \nu_2 = 1, \ \delta_2 = 1/2, \ \nu_3 = 1/2, \ r = s = 1, \tag{2.35}$$

provided that

$$d^2 = 1, \quad 38\gamma_3^2 = 25\mu_1 + 13\mu_2 + 19\mu_3 \tag{2.36}$$

Now, taking $e_1 = 1$ in (2.18) one has

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}_1, \tag{2.37}$$

$$\hat{\varphi}_1 = -\frac{4}{\beta_0}\arctan\{d \ \exp[\gamma_3(x - vt)]\}. \tag{2.38}$$

This is a sine-Gordon 1-antisoliton associated to both fields $\varphi_{1,2}$ in the particular case when they are equal to each other, and it corresponds to the relations (2.20). It possesses a mass $M_3 = \frac{8\gamma_3}{\beta_0^2}$.

In view of the symmetry (2.12) we are able to write

$$d^2 = 1, \quad 38\gamma_4^2 = 25\mu_2 + 13\mu_1 + 19\mu_3, \tag{2.39}$$

and then on has another soliton of this type

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}_2, \tag{2.40}$$

$$\hat{\varphi}_2 = -\frac{4}{\beta_0}\arctan\{d \ \exp[\gamma_4(x - vt)]\}. \tag{2.41}$$

It possesses a mass $M_4 = \frac{8\gamma_4}{\beta_0^2}$ and corresponds to the relations (2.20). This 1-antisoliton is of the type shown in figure 3.

The GSG system (2.10)–(2.11) reduces to the usual SG equation for each choice of the parameters (2.25), (2.30) and (2.35), respectively. Then, the $n-$soliton solutions in each case can be constructed as in the ordinary sine-Gordon model by taking appropriate tau functions in (2.14)–(2.15).

In view of the symmetry (2.13) one can be able to construct the *solitons* corresponding to the antisoliton solutions (2.27), (2.32), (2.38) and (2.41) simply by changing their signs $\varphi_a \to -\varphi_a$.

A modified model with rich soliton dynamics is the so-called stepwise sine-Gordon model in which the system parameter depends on the sign of the SG field [24]. It would be interesting to consider the above GSG model along the lines of this reference.

2.4 Mass splitting of solitons

It is interesting to write some relations among the various soliton masses

$$M_3^2 = \frac{1}{76}(109M_2^2 + 5M_1^2); \quad M_4^2 = \frac{1}{76}(109M_1^2 + 5M_2^2); \tag{2.42}$$

If $\mu_1 = \mu_2$ then we have the degeneracy $M_1 = M_2$, and $M_3 = M_4 = \sqrt{3/2}M_1$. Notice that if $M_1 \neq M_2$ then $M_3 < M_1 + M_2$ and $M_4 < M_1 + M_2$, and the third and fourth solitons are stable in the sense that energy is required to dissociate them.

2.5 Kinks of the reduced two-frequency sine-Gordon model

In the system (2.10)–(2.11) we perform the following reduction $\varphi \equiv \varphi_1 = \varphi_2$ such that

$$\Phi_1 = \Phi_2, \ \ \Phi_3 = q\,\Phi_1, \tag{2.43}$$

with q being a real number. Therefore, using the constraint (2.4) one can deduce the relationships

$$\delta_1 = \frac{q}{2}, \ \ \delta = 1. \tag{2.44}$$

Moreover, for consistency of the system of equations (2.10)–(2.11) we have to impose the relationships

$$\nu_1\mu_1\Delta_{11} + \nu_2\mu_2\Delta_{12} = \nu_1\mu_1\Delta_{21} + \nu_2\mu_2\Delta_{22}, \tag{2.45}$$

$$\Delta_{13} = \Delta_{23}. \tag{2.46}$$

These relations imply

$$\delta^2 = 1, \ \ \mu_1 = \delta\,\mu_2 \tag{2.47}$$

.

Taking into account the relations (2.44) and (2.47) together with (2.5) we get

$$\mu_1 = \mu_2, \ \ \delta = 1, \ \ \nu_1 = \nu_2, \ \ \nu_3 = \frac{q}{2}\nu_1, \ \ r = s = 1. \tag{2.48}$$

Thus the system of eqs. (2.10)–(2.11) reduce to

$$\partial^2\Phi = -\frac{\mu_1}{\nu_1}\sin(\nu_1\Phi) - \frac{\mu_3\delta_1}{\nu_1}\sin(q\,\nu_1\Phi), \ \ \Phi \equiv \beta_0\varphi. \tag{2.49}$$

This is the so-called *two-frequency (double) sine-Gordon* model (DSG) and it has been the subject of much interest in the last decades, from the mathematical and physical points of view. It encounters many interesting physical applications, such as to the study of massive Schwinger model (two-dimensional quantum electrodynam- ics) and a generalized Ashkin-Teller model (a quantum spin system). A further potentially interesting application of the two-(and multi-)frequency sine-Gordon model is for ultra-short optical pulses propagating in resonant degenerate medium (see e.g. [3, 4, 19, 20]).

If the parameter q satisfies

$$q = \frac{n}{m} \ \in \ \mathbb{Q} \tag{2.50}$$

with m, n being two relative prime positive integers, then the potential $\frac{\mu_1}{\nu_1^2}(1 - \cos(\nu_1\Phi)) + \frac{\mu_3}{2\nu_1^2}(1 - \cos(q\nu_1\Phi))$ associated to the model (2.49) is periodic with period

$$\frac{2\pi}{\nu_1}m = \frac{2\pi}{q\,\nu_1}n. \tag{2.51}$$

As mentioned above the theory (2.49) possesses topological excitations. The fundamental topological excitations degenerates in the $\mu_1 = 0$ limit to an $n-$soliton state of the relevant sine-Gordon model and similarly in the limit $\mu_3 = 0$ it will be an m-soliton state. For general values of the parameters μ_1, μ_3, δ_1, ν_1 the solitons are in some sense "confined" inside the topological excitations which become in this form some composite objects. On the other hand, if $q \notin \mathbb{Q}$ then the potential is not periodic, so, there are no topologically charged excitations and the solitons are completely confined [1, 2].

The model (2.49) in the limit $\mu_1 = 0$ reduces to

$$\partial^2 \varphi = -\frac{\mu_3 q}{2\nu_1 \beta_0} \sin(q\nu_1 \beta_0 \varphi). \tag{2.52}$$

For later discussion we record here the mass of the soliton associated to this equation,

$$M_{\mu_3} = \frac{8}{(q\nu_1 \beta_0)^2} \sqrt{q^2 \mu_3/2}. \tag{2.53}$$

Correspondingly in the limit $\mu_3 = 0$ one has

$$\partial^2 \varphi = -\frac{mu_1}{\nu_1 \beta_0} \sin(\nu_1 \beta_0 \varphi) \tag{2.54}$$

with associated soliton mass

$$M_{\mu_1} = \frac{8}{(\nu_1 \beta_0)^2} \sqrt{\mu_1} \tag{2.55}$$

Notice that other possibilities to perform the reduction of type (2.43) encounter some inconsistencies, e.g. the attempt to implement the reduction $\Phi_1 = \Phi_3$, $\Phi_2 = q'\, \Phi_1$ implies $\delta_{1,2}^2 < 0$ which is a contradiction since $\delta_{1,2}$ are real numbers by definition. The same inconsistency occurs when one tries to reduce the $sl(3,\mathbb{C})$ GSG model to a three-frequency SG model. We expect that the three and higher frequency models [25] will be related to $sl(N,\mathbb{C})$, $N \geq 4$, GSG models.

In the following we will provide some kink solutions for particular set of parameters. Consider

$$\nu_1 = 1/2, \ \delta_1 = \delta_2 = 1, \ \nu_2 = 1/2, \ \nu_3 = 1/2 \ \text{and} \ q = 2, n = 2, m = 1 \tag{2.56}$$

which satisfy (2.48) and (2.50), respectively. This set of parameters provide the so-called *double sine-Gordon model* (DSG). Its potential $-[4\mu_1(\cos\frac{\Phi}{2} - 1) + 2\mu_3(\cos\Phi - 1)]$ has period 4π and has extrema at $\Phi = 2\pi p_1$, and $\Phi = 4\pi p_2 \pm 2\cos^{-1}[1 - |\mu_1/(2\mu_3)|]$ with $p_1, p_2 \in \mathbb{Z}$; the second extrema exists only if $|\mu_1/(2\mu_3)| < 1$. From the mathematical point of view the DSG model belongs to a class of theories with partial integrability [26]. Depending on the values of the parameters β_0, μ_1, μ_3 the quantum field theory version of the DSG model presents a variety of physical effects, such as the decay of the false vacuum, a phase transition, confinement of the kinks and the resonance phenomenon due to unstable bound states of excited kink-antikink states (see [4] and references therein). The semi-classical spectrum of neutral particles in the DSG theory is investigated in [27].

Interestingly the functions[1]

$$\tau_0 = 1 + i\,d\exp[\gamma(x - vt)] + h\exp[2\gamma(x - vt)],$$
$$\tau_1 = 1 - i\,d\exp[\gamma(x - vt)] + h\exp[2\gamma(x - vt)], \qquad (2.57)$$

satisfy the equation (2.49) for the parameters (2.56) provided

$$e^{-i\Phi/2} = \tau_1/\tau_0 \qquad (2.58)$$

$$\gamma^2 = \mu_1 + 2\mu_3, \quad h = -\frac{\mu_1}{4}, \quad e_1 = -1 \qquad (2.59)$$

The general solution of this type can be written as

$$\Phi := 4\arctan\left[\frac{1}{d}\,\frac{1 + h\exp[2\gamma(x - vt)]}{\exp[\gamma(x - vt)]}\right] \qquad (2.60)$$

2.5.1 DSG kink ($h < 0$, $\mu_i > 0$)

For the choice of parameters $h < 0$, $\mu_i > 0$ in (2.59) the equation (2.60) provides

$$\varphi = \frac{4}{\beta_0}\arctan\left[\frac{-2|h|^{1/2}}{d}\,\sinh[\gamma_K\,(x - vt) + a_0]\right], \quad \gamma_K \equiv \pm\sqrt{\mu_1 + 2\mu_3}, \quad (2.61)$$

$$a_0 = \frac{1}{2}\ln|h|.$$

This is the DSG 1-kink solution with mass

$$M_K = \frac{16}{\beta_0^2}\gamma_K\left[1 + \frac{\mu_1}{\sqrt{2\mu_3(\mu_1 + 2\mu_3)}}\ln(\frac{\sqrt{\mu_1 + 2\mu_3} + \sqrt{2\mu_3}}{\sqrt{\mu_1}})\right]. \qquad (2.62)$$

Notice that in the limit $\mu_1 \to 0$ the kink mass becomes $M_K = \frac{16}{\beta_0^2}\sqrt{2\mu_3}$, which is twice the soliton mass (2.53) of the model (2.52) for the parameters $\nu_1 = 1/2$, $q = 2$. Similarly, in the limit $\mu_3 \to 0$ the kink mass becomes $\frac{8}{(\beta_0/2)^2}\sqrt{\mu_1}$, which is the soliton mass (2.55) of the model (2.54) for $\nu_1 = 1/2$, $q = 2$; thus in this case the coupling constant is $\beta_0/2$. As discussed above these solitons get in some sense "confined" inside the kink if the parameters satisfy $\mu_i \neq 0$. The 1-antikink is plotted in figure 4.
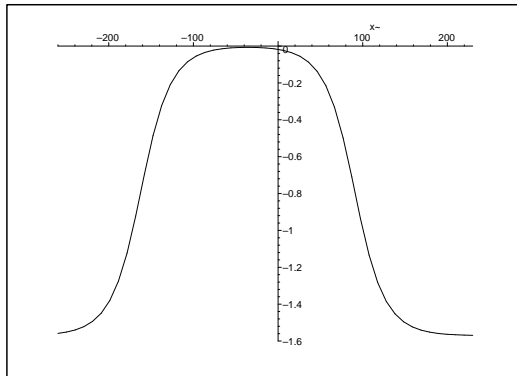
2.5.2 Bounce-like solution ($h > 0$, $\mu_1 < 0$)

For the parameters $h > 0$, $\mu_1 < 0$ one gets from (2.60)

$$\varphi := \frac{4}{\beta_0}\arctan\left[\frac{2h^{1/2}}{d}\,\cosh[\gamma'(x - vt) + a_0']\right], \quad \gamma' = 2\mu_3 - |\mu_1|, \quad a_0' = \frac{1}{2}\ln h \quad (2.63)$$

This is the bounce-like solution and interpolates between the two vacuum values 2π and $4\pi - 2\arccos(1 - |\mu_1/2\mu_3|)$ and then it comes back. Since 2π is a false vacuum position this solution is not related to any stable particle in the quantum theory [4]. In figure 2 we plot this profile.

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[1]These functions are obtained by adding the term $\exp[2\gamma(x - vt)]$ to the relevant tau functions for one solitons used above. This procedure adds a new method of solving DSG which deserve further study. The multi-frequency SG equations can be solved through the Jacobi elliptic function expansion method, see e.g. [28].

Figure 2: Bounce-like solution $(\frac{\beta_0}{4}\varphi)$ plotted for $\mu_2 = -0.0000001$, $\mu_3 = 0.001$, $d = -2$.

3. Classical GSG as a reduced affine Toda model coupled to matter

In this section we provide the algebraic construction of the $sl(3,\mathbb{C})$ conformal affine Toda model coupled to matter fields (CATM) and closely follows refs. [6, 22, 29] but the reduction process to arrive at the classical GSG model is new. The CATM model is a two-dimensional field theory involving four scalar fields and six Dirac spinors. The interactions among the fields are as follows: 1) in the scalars equations of motion there are the coupling of bilinears in the spinors to exponentials of the scalars. 2) Some of the equations of motion for the spinors have certain bilinear terms in the spinors themselves. That fact makes it difficult to find a local Lagrangian for the theory. Nevertheless, the model presents a lot of symmetries. It is conformally invariant, possesses local gauge symmetries as well as vector and axial conserved currents bilinear in the spinors. One of the most remarkable properties of the model is that it presents an equivalence between a U(1) vector conserved current, bilinear in the spinors, and a topological currents depending only on the first derivative of some scalars. This property allow us to implement a bag model like confinement mechanism resembling what one expects to happen in QCD. The model possesses a zero curvature representation based on the $\hat{sl}3(C)$ affine Kac Moody algebra. It constitutes a particular example of the so-called conformal affine Toda models coupled to matter fields which have been introduced in [29]. The corresponding model associated to $\hat{sl}2(C)$ has been studied in [31] where it was shown, using bosonization techniques, that the equivalence between the currents holds true at the quantum level and so the confinement mechanism does take place in the quantum theory.

The off-critical affine Toda model coupled to matter (ATM) is defined by gauge fixing the conformal symmetry [6, 30]. The previous treatments of the $sl(3,\mathbb{C})$ ATM model used the symplectic and on-shell decoupling methods to unravel the classical generalized sine-Gordon (cGSG) and generalized massive Thirring (GMT) dual theories describing the strong/weag coupling sectors of the ATM model [5, 6, 30]. As mentioned above the ATM model describes some scalars coupled to spinor (Dirac) fields in which the system of equations of motion has a local gauge symmetry. Conveniently gauge fixing the local symmetry by setting some spinor bilinears to constants we are able to decouple the scalar

(Toda) fields from the spinors, the final result is a direct construction of the classical generalized sine-Gordon model (cGSG) involving only the scalar fields. In the spinor sector we are left with a system of equations in which the Dirac fields couple to the cGSG fields. Another instance in which the quantum version of the generalized sine-Gordon theory arises is in the process of bosonization of the generalized massive Thirring model (GMT), which is a multiflavor extension of the usual massive Thirring model such that, apart from the usual current-current self-interaction for each flavor, it presents current-current interactions terms among the various U(1) flavor currents [7].

The zero curvature condition (see (A.1) and the appendix) gives the following equations of motion for the CATM model [29]

$$\frac{\partial^2 \theta_a}{4i\,e^\eta} = m_1[e^{\eta - i\phi_a}\widetilde{\psi}_R^l \psi_L^l + e^{i\phi_a}\widetilde{\psi}_L^l \psi_R^l] + m_3[e^{-i\phi_3}\widetilde{\psi}_R^3 \psi_L^3 + e^{\eta + i\phi_3}\widetilde{\psi}_L^3 \psi_R^3]; \quad a = 1, 2 \tag{3.1}$$

$$-\frac{\partial^2 \widetilde{\nu}}{4} = im_1 e^{2\eta - \phi_1}\widetilde{\psi}_R^1 \psi_L^1 + im_2 e^{2\eta - \phi_2}\widetilde{\psi}_R^2 \psi_L^2 + im_3 e^{\eta - \phi_3}\widetilde{\psi}_R^3 \psi_L^3 + \mathbf{m}^2 e^{3\eta}, \tag{3.2}$$

$$-2\partial_+ \psi_L^1 = m_1 e^{\eta + i\phi_1}\psi_R^1, \qquad -2\partial_+ \psi_L^2 = m_2 e^{\eta + i\phi_2}\psi_R^2, \tag{3.3}$$

$$2\partial_- \psi_R^1 = m_1 e^{2\eta - i\phi_1}\psi_L^1 + 2i\left(\frac{m_2 m_3}{im_1}\right)^{1/2} e^\eta(-\psi_R^3 \widetilde{\psi}_L^2 e^{i\phi_2} - \widetilde{\psi}_R^2 \psi_L^3 e^{-i\phi_3}), \tag{3.4}$$

$$2\partial_- \psi_R^2 = m_2 e^{2\eta - i\phi_2}\psi_L^2 + 2i\left(\frac{m_1 m_3}{im_2}\right)^{1/2} e^\eta(\psi_R^3 \widetilde{\psi}_L^1 e^{i\phi_1} + \widetilde{\psi}_R^1 \psi_L^3 e^{-i\phi_3}), \tag{3.5}$$

$$-2\partial_+ \psi_L^3 = m_3 e^{2\eta + i\phi_3}\psi_R^3 + 2i\left(\frac{m_1 m_2}{im_3}\right)^{1/2} e^\eta(-\psi_L^1 \psi_R^2 e^{i\phi_2} + \psi_L^2 \psi_R^1 e^{i\phi_1}), \tag{3.6}$$

$$2\partial_- \psi_R^3 = m_3 e^{\eta - i\phi_3}\psi_L^3, \qquad 2\partial_- \widetilde{\psi}_R^1 = m_1 e^{\eta + i\phi_1}\widetilde{\psi}_L^1, \tag{3.7}$$

$$-2\partial_+ \widetilde{\psi}_L^1 = m_1 e^{2\eta - i\phi_1}\widetilde{\psi}_R^1 + 2i\left(\frac{m_2 m_3}{im_1}\right)^{1/2} e^\eta(-\psi_L^2 \widetilde{\psi}_R^3 e^{-i\phi_3} - \widetilde{\psi}_L^3 \psi_R^2 e^{i\phi_2}), \tag{3.8}$$

$$-2\partial_+ \widetilde{\psi}_L^2 = m_2 e^{2\eta - i\phi_2}\widetilde{\psi}_R^2 + 2i\left(\frac{m_1 m_3}{im_2}\right)^{1/2} e^\eta(\psi_L^1 \widetilde{\psi}_R^3 e^{-i\phi_3} + \widetilde{\psi}_L^3 \psi_R^1 e^{i\phi_1}), \tag{3.9}$$

$$2\partial_- \widetilde{\psi}_R^2 = m_2 e^{\eta + i\phi_2}\widetilde{\psi}_L^2, \qquad -2\partial_+ \widetilde{\psi}_L^3 = m_3 e^{\eta - i\phi_3}\widetilde{\psi}_R^3, \tag{3.10}$$

$$2\partial_- \widetilde{\psi}_R^3 = m_3 e^{2\eta + i\phi_3}\widetilde{\psi}_L^3 + 2i\left(\frac{m_1 m_2}{im_3}\right)^{1/2} e^\eta(\widetilde{\psi}_R^1 \widetilde{\psi}_L^2 e^{i\phi_2} - \widetilde{\psi}_R^2 \widetilde{\psi}_L^1 e^{i\phi_1}), \tag{3.11}$$

$$\partial^2 \eta = 0, \tag{3.12}$$

where $\phi_1 \equiv 2\theta_1 - \theta_2$, $\phi_2 \equiv 2\theta_2 - \theta_1$, $\phi_3 \equiv \theta_1 + \theta_2$. Therefore, one has

$$\phi_3 = \phi_1 + \phi_2 \tag{3.13}$$

The θ fields are considered to be in general complex fields. In order to define the classical generalized sine-Gordon model we will consider these fields to be real.

Apart from the *conformal invariance* the above equations exhibit the $(\mathrm{U}(1)_L)^2 \otimes (\mathrm{U}(1)_R)^2$ *left-right local gauge symmetry*

$$\theta_a \rightarrow \theta_a + \xi_+^a(x_+) + \xi_-^a(x_-), \quad a = 1, 2 \tag{3.14}$$

$$\widetilde{\nu} \rightarrow \widetilde{\nu}\,; \qquad \eta \rightarrow \eta \tag{3.15}$$

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$$\psi^i \rightarrow e^{i(1+\gamma_5)\Xi^i_+(x_+)+i(1-\gamma_5)\Xi^i_-(x_-)}\,\psi^i, \tag{3.16}$$

$$\widetilde{\psi}^i \rightarrow e^{-i(1+\gamma_5)(\Xi^i_+)(x_+)-i(1-\gamma_5)(\Xi^i_-)(x_-)}\,\widetilde{\psi}^i, \quad i=1,2,3; \tag{3.17}$$

$$\Xi^1_\pm \equiv \pm\xi^2_\pm \mp 2\xi^1_\pm, \ \Xi^2_\pm \equiv \pm\xi^1_\pm \mp 2\xi^2_\pm, \ \Xi^3_\pm \equiv \Xi^1_\pm + \Xi^2_\pm.$$

One can get global symmetries for $\xi^a_\pm = \mp\xi^a_\mp = $ constants. For a model defined by a Lagrangian these would imply the presence of two vector and two chiral conserved currents. However, it was found only half of such currents [22]. This is a consequence of the lack of a Lagrangian description for the $sl(3)^{(1)}$ CATM in terms of the B and $F^\pm$ fields (see appendix). So, the vector current

$$J^\mu = \sum_{j=1}^3 m_j \bar{\psi}^j \gamma^\mu \psi^j \tag{3.18}$$

and the chiral current

$$J^{5\,\mu} = \sum_{j=1}^3 m_j \bar{\psi}^j \gamma^\mu \gamma_5 \psi^j + 2\partial_\mu(m_1\theta_1 + m_2\theta_2) \tag{3.19}$$

are conserved

$$\partial_\mu J^\mu = 0, \quad \partial_\mu J^{5\,\mu} = 0 \tag{3.20}$$

The conformal symmetry is gauge fixed by setting [30]

$$\eta = \text{const.} \tag{3.21}$$

The off-critical ATM model obtained in this way exhibits the vector and topological currents equivalence [29, 30]

$$\sum_{j=1}^3 m_j \bar{\psi}^j \gamma^\mu \psi^j \equiv \epsilon^{\mu\nu}\partial_\nu(m_1\theta_1 + m_2\theta_2), \quad m_3 = m_1 + m_2, \ m_i > 0. \tag{3.22}$$

In the next steps we implement the reduction process to get the cGSG model through a gauge fixing of the ATM theory. The local symmetries (3.14)–(3.17) can be gauge fixed through

$$i\bar{\psi}^j \psi^j = iA_j = \text{const.}; \quad \bar{\psi}^j \gamma_5 \psi^j = 0. \tag{3.23}$$

From the gauge fixing (3.23) one can write the following bilinears

$$\widetilde{\psi}^j_R \psi^j_L + \widetilde{\psi}^j_L \psi^j_R = 0, \quad j = 1, 2, 3; \tag{3.24}$$

so, the eqs. (3.23) effectively comprises three gauge fixing conditions.

It can be directly verified that the gauge fixing (3.23) preserves the currents conservation laws (3.20), i.e. from the equations of motion (3.1)–(3.12) and the gauge fixing (3.23) together with (3.21) it is possible to obtain the currents conservation laws (3.20).

Taking into account the constraints (3.23) in the scalar sector, eqs. (3.1), we arrive at the following system of equations (set $\eta = 0$)

$$\partial^2\theta_1 = M_\psi^1 \sin(2\theta_1 - \theta_2) + M_\psi^3 \sin(\theta_1 + \theta_2), \tag{3.25}$$

$$\partial^2\theta_2 = M_\psi^2 \sin(2\theta_2 - \theta_1) + M_\psi^3 \sin(\theta_1 + \theta_2), \quad M_\psi^i \equiv 4A_i\, m_i, \quad i = 1,2,3. \tag{3.26}$$

Define the fields φ_1, φ_2 as

$$\varphi_1 \equiv a\theta_1 + b\theta_2, \quad a = \frac{4\nu_2 - \nu_1}{3\beta_0\nu_1\nu_2}, \quad d = \frac{4\nu_1 - \nu_2}{3\beta_0\nu_1\nu_2} \tag{3.27}$$

$$\varphi_2 \equiv c\theta_1 + d\theta_2, \quad b = -c = \frac{2(\nu_1 - \nu_2)}{3\beta_0\nu_1\nu_2}, \quad \nu_1,\nu_2 \in \mathbb{R} \tag{3.28}$$

Then, the system of equations (3.25)–(3.26) written in terms of the fields $\varphi_{1,2}$ becomes

$$\partial^2\varphi_1 = aM_\psi^1 \sin[\beta_0\nu_1(2\varphi_1 - \varphi_2)] + bM_\psi^2 \sin[\beta_0\nu_2(2\varphi_2 - \varphi_1)] + \\ (a+b)M_\psi^3 \sin\beta_0[(2\nu_1 - \nu_2)\varphi_1 + (2\nu_2 - \nu_1)\varphi_2], \tag{3.29}$$

$$\partial^2\varphi_2 = cM_\psi^1 \sin[\beta_0\nu_1(2\varphi_1 - \varphi_2)] + dM_\psi^2 \sin[\beta_0\nu_2(2\varphi_2 - \varphi_1)] + \\ (c+d)M_\psi^3 \sin\beta_0[(2\nu_1 - \nu_2)\varphi_1 + (2\nu_2 - \nu_1)\varphi_2] \tag{3.30}$$

The system of equations above considered for real fields $\varphi_{1,2}$ as well as for real parameters $M_\psi^i, a, b, c, d, \beta_0$ defines the *classical generalized sine-Gordon model* (cGSG). Notice that this classical version of the GSG model derived from the ATM theory is a submodel of the GSG model (2.10)–(2.11), defined in section 2, for the particular parameter values $r = \frac{2\nu_1 - \nu_2}{\nu_3}$, $s = \frac{2\nu_2 - \nu_1}{\nu_3}$ and the convenient identifications of the parameters in the coefficients of the sine functions of the both models.

The following reduced models can be obtained from the system (3.29)–(3.30):

(i)*SG submodels*

    (i.1) For $\nu_2 = 2\nu_1$ one has $M_\psi^1 = M_\psi^2$ and the system $\varphi_2 = 0$, $\partial^2\varphi_1 = M_\psi^1 \frac{3\nu_1}{\beta_0}\sin\beta_0 2\nu_1\varphi_1$.

    (i.2) For $\nu_1 = 2\nu_2$ one has $M_\psi^1 = M_\psi^2$ and the system $\varphi_1 = 0$, $\partial^2\varphi_2 = M_\psi^2 \frac{3\nu_2}{\beta_0}\sin\beta_0 2\nu_2\varphi_2$.

    (i.3) For $\nu_2 = \nu_1 \equiv \nu$ and $\varphi_1 = \varphi_2 \equiv \hat\varphi_A, (A = 1,2)$, one gets the sub-models

        (i.3a) $M_\psi^1 = M_\psi^2, M_\psi^3 = 0$, $\partial^2\hat\varphi_1 = aM_\psi^1\sin\beta_0\nu\hat\varphi_1$,

        (i.3b) $M_\psi^1 = M_\psi^2 = 0$, $\quad\partial^2\hat\varphi_2 = aM_\psi^3 \sin\beta_0\nu\hat\varphi_2$.

(ii) *DSG sub-model*

    For $\nu_1 = \nu_2$ and $M_\psi^1 = M_\psi^2$ one gets the sub-model $\varphi_1 = \varphi_2 \equiv \varphi$, $\partial^2\varphi = aM_\psi^1\sin\beta_0\nu_1\varphi + aM_\psi^3\sin2\beta_0\nu_1\varphi$.

The sub-models i.1)-i.2) each one contains the ordinary sine-Gordon model (SG) and they were considered in the subsections 2.1 and 2.2, respectively; the sub-model i.3) supports two SG models with different soliton masses which must correspond to the construction in subsection 2.3; and the ii) case defines the double sine-Gordon model (DSG) studied in subsection 2.5. Other meaningful reductions are possible arriving at either SG

or DSG model. Notice that the reductions above are particular cases of the sub-models in subsections 2.1, 2.2, 2.3 and 2.5, respectively, for relevant parameter identifications.

In view of the gauge fixing (3.23) the spinor sector can be parameterized conveniently as

$$\begin{pmatrix} \psi_R^j \\ \psi_L^j \end{pmatrix} = \begin{pmatrix} \sqrt{A_j/2}\, u_j \\ i\sqrt{A_j/2}\, \frac{1}{v_j} \end{pmatrix}; \quad \begin{pmatrix} \tilde{\psi}_R^j \\ \tilde{\psi}_L^j \end{pmatrix} = \begin{pmatrix} \sqrt{A_j/2}\, v_j \\ -i\sqrt{A_j/2}\, \frac{1}{u_j} \end{pmatrix}. \tag{3.31}$$

Therefore, in order to find the spinor field solutions one can solve the eqs. (3.3)–(3.11) for the fields u_j, v_j for each solution given for the cGSG fields $\varphi_{1,2}$ of the system (3.29)–(3.30).

In the context of the Hirota and dressing transformation methods one can construct the soliton solutions of the off-critical ATM model using the fields parameterizations in terms of the tau functions [22]

$$e^{-i\theta_1} = \frac{\hat{\tau}_1}{\hat{\tau}_0}, \qquad e^{-i\theta_2} = \frac{\hat{\tau}_2}{\hat{\tau}_0}, \qquad e^{-\left(\tilde{\nu} + \frac{1}{12}\sum_{i=1}^3 m_i^2\, x^+ x^-\right)} = \hat{\tau}_0. \tag{3.32}$$

$$\psi_R^1 = \frac{m_3}{m_1}\frac{\tau_{R,(2)}^{\alpha_1}}{\hat{\tau}_2} - \frac{m_2}{m_1}\frac{\tau_{R,(0)}^{\alpha_1}}{\hat{\tau}_0}, \qquad\qquad \psi_L^1 = -\frac{\tau_L^{\alpha_1}}{\hat{\tau}_1}, \tag{3.33}$$

$$\psi_R^2 = \frac{m_3}{m_2}\frac{\tau_{R,(1)}^{\alpha_2}}{\hat{\tau}_1} - \frac{m_1}{m_2}\frac{\tau_{R,(0)}^{\alpha_2}}{\hat{\tau}_0}, \qquad\qquad \psi_L^2 = -\frac{\tau_L^{\alpha_2}}{\hat{\tau}_2}, \tag{3.34}$$

$$\psi_R^3 = \frac{\tau_R^{\alpha_3}}{\hat{\tau}_0}, \qquad\qquad \psi_L^3 = \frac{m_2}{m_3}\frac{\tau_{L,(1)}^{\alpha_3}}{\hat{\tau}_1} + \frac{m_1}{m_3}\frac{\tau_{L,(2)}^{\alpha_3}}{\hat{\tau}_2}, \tag{3.35}$$

$$\tilde{\psi}_R^1 = -\frac{\tilde{\tau}_R^{\alpha_1}}{\hat{\tau}_1}, \qquad\qquad \tilde{\psi}_L^1 = \frac{m_2}{m_1}\frac{\tilde{\tau}_{L,(0)}^{\alpha_1}}{\hat{\tau}_0} - \frac{m_3}{m_1}\frac{\tilde{\tau}_{L,(2)}^{\alpha_1}}{\hat{\tau}_2}, \tag{3.36}$$

$$\tilde{\psi}_R^2 = -\frac{\tilde{\tau}_R^{\alpha_2}}{\hat{\tau}_2}, \qquad\qquad \tilde{\psi}_L^2 = \frac{m_1}{m_2}\frac{\tilde{\tau}_{L,(0)}^{\alpha_2}}{\hat{\tau}_0} - \frac{m_3}{m_2}\frac{\tilde{\tau}_{L,(1)}^{\alpha_2}}{\hat{\tau}_1}, \tag{3.37}$$

$$\tilde{\psi}_R^3 = -\frac{m_2}{m_3}\frac{\tilde{\tau}_{R,(1)}^{\alpha_3}}{\hat{\tau}_1} - \frac{m_1}{m_3}\frac{\tilde{\tau}_{R,(2)}^{\alpha_3}}{\hat{\tau}_2}, \qquad\qquad \tilde{\psi}_L^3 = -\frac{\tilde{\tau}_L^{\alpha_3}}{\hat{\tau}_0}, \tag{3.38}$$

These tau functions must satisfy the relationships

$$\tilde{\tau}_R^{\alpha_2}\,\tau_R^{\alpha_3} + \hat{\tau}_2\,\tau_{R,(0)}^{\alpha_1} - \hat{\tau}_0\,\tau_{R,(2)}^{\alpha_1} = 0, \qquad \tilde{\tau}_R^{\alpha_1}\,\tau_R^{\alpha_3} - \hat{\tau}_1\,\tau_{R,(0)}^{\alpha_2} + \hat{\tau}_0\,\tau_{R,(1)}^{\alpha_2} = 0, \tag{3.39}$$

$$\tilde{\tau}_R^{\alpha_1}\,\tilde{\tau}_R^{\alpha_2} - \hat{\tau}_2\,\tilde{\tau}_{R,(1)}^{\alpha_3} + \hat{\tau}_1\,\tilde{\tau}_{R,(2)}^{\alpha_3} = 0, \qquad \tau_L^{\alpha_2}\,\tilde{\tau}_L^{\alpha_3} - \hat{\tau}_0\,\tilde{\tau}_{L,(2)}^{\alpha_1} + \hat{\tau}_2\,\tilde{\tau}_{L,(0)}^{\alpha_1} = 0, \tag{3.40}$$

$$\tau_L^{\alpha_1}\,\tilde{\tau}_L^{\alpha_3} + \hat{\tau}_0\,\tilde{\tau}_{L,(1)}^{\alpha_2} - \hat{\tau}_1\,\tilde{\tau}_{L,(0)}^{\alpha_2} = 0, \qquad \tau_L^{\alpha_1}\,\tau_L^{\alpha_2} + \hat{\tau}_2\,\tau_{L,(1)}^{\alpha_3} - \hat{\tau}_1\,\tau_{L,(2)}^{\alpha_3} = 0. \tag{3.41}$$

The Hirota method requires the determination of the equations of motion satisfied by the *tau* functions. They are determined by substituting the relations between the fields and tau functions (3.32)–(3.38) into the equations of motion (3.1)–(3.11). Except for η, one has 15 fields in the model (3.1)–(3.12) (three scalars and six two-component spinors). However, in (3.32)–(3.38) we have defined 21 tau-functions. The algebraic relations (3.39)–(3.41) which arise in the framework of the dressing method provide the six equations missing. Due to the local gauge invariance (3.14)–(3.17) from the 15 fields of the model mentioned above only 9 of them describe the physical degrees of freedom which will be related to the physical soliton spectrum of the theory as discussed below.

The dressing transformation method does not excite the field η if one starts from a solution where it vanishes. Moreover, it has been shown that the soliton type solutions are in the orbit of the vacuum $\eta = 0$ [29].

3.1 Physical solitons and kinks of the ATM model

The main feature of the one 'solitons' constructed in [22] is that for each positive root of $sl(3)$ there corresponds one soliton species associated to the fields ϕ_1, ϕ_2, ϕ_3, respectively. The relevant solutions for the spinor fields together with the 1-'solitons' satisfy the relationship (3.22). The class of 2-'soliton' solutions of $sl(3)$ ATM obtained in [22] behave as follows:

(i) they are given by 6 species associated to the pair (α_i, α_j), $i \leq j$; $i, j = 1, 2, 3$; where the α's are the positive roots of $sl(3)$ Lie algebra. Each species (α_i, α_i) solves the $sl(2)$ ATM submodel.[2]

(ii) they satisfy the U(1) vector and topological currents equivalence 3.22. However, the possible kink type solutions associated in a *non-local* way to the spinor bilinears and the relevant gauge fixing of the local symmetry (3.14)–(3.17) have not been discussed in the literature.

In order to consider the physical spectrum of solitons and study their properties, such as masses and scattering time delays, it is mandatory to take into account these questions which are related to the counting of the true physical degrees of freedom of the theory. Therefore, one must consider the possible soliton type solutions associated to each spinor bilinear. The relation between this type of 'solitons', say $\hat{\phi}_j$, and their relevant fermion bilinears must be non-local as suggested by the equivalence equation (3.22). So, we may have soliton solutions of type

$$\hat{\phi}_j = \int^x dx' \, \bar{\psi}^j \gamma^0 \psi^j, \quad j = 1, 2, 3 \tag{3.42}$$

At this stage one is able to enumerate the physical 1-soliton (1-antisoliton) spectrum associated to the gauge fixed ATM model. In fact, we have three 'kinks' and their corresponding 'anti-kinks' associated to the fields ϕ_i (i=1,2,3), and three kink and antikink pairs of type $\hat{\phi}_j$, $j = 1, 2, 3$. Thus, we have six kink and their relevant antikink solutions, but in order to record the physical soliton and anti-soliton excitations one must take into account the four constraints (3.13) and (3.24). Therefore, we expect to find four pairs of soliton and anti-soliton physical excitations in the spectrum. This feature is nicely reproduced in the cGSG sector of the ATM model; in fact, in the last section we were able to write four usual sine-Gordon models as possible reductions of the cGSG model. Namely, 1-soliton (1-antisoliton) associated to the fields φ_1, φ_2, respectively (subsections 2.1 and 2.2) and 1-solitons (1-antisolitons) associated to the fields $\hat{\varphi}_A$, $A = 1, 2$, respectively (subsection 2.3),

---

[2]$sl(2)$ ATM gauge unfixed 2−'solitons' satisfy an analogous eq. to 3.22. Moreover, for φ real and $\widetilde{\psi} = \pm(\psi)^*$ one has, soliton-soliton SS, SS bounds and no $S\bar{S}$ (S =soliton, $\bar{S}$=anti-soliton) bounds [31] associated to the field φ.

thus, providing four pairs of 1- soliton/1-antisoliton. In the 2-kink (2-antikink) sector a similar argument will provide us ten physical 2-solitons and their relevant 2-antisoliton excitations, i.e. six pairs of 2-kink and 2-antikink solutions of type ϕ and $\hat{\phi}$, respectively, which give twenty four excitations, and taking into account the constraints (3.13) and (3.24) we are left with ten pairs of 2-solitons and 2-antisolitons. In fact, these ten 2-solitons correspond to the pairs we can form with the four species of 1-solitons found in the last section, in all possible ways. The same argument holds for the corresponding ten 2-antisolitons.

In this way the system (3.29)–(3.30) gives rise to a richer (anti)soliton spectrum and dynamics than the θ_a field 'soliton' type solutions of the gauge unfixed model (3.1)–(3.11) found in [22]. Regarding this issue let us notice that in the procedure followed in ref. [22] the local symmetry (3.14)–(3.17) and the relevant gauge fixing has not been considered explicitly, therefore their 'solitons' do not correspond to the GSG solitons obtained above.

Notice that the tau functions in section 2 possess the function $\gamma(x - vt)$ in their exponents, whereas the corresponding ones in the ATM theory have two times this function [22, 31]. This fact is reflected in the GSG soliton solutions which are two times the relevant solutions of the ATM model. It has been observed already in the $sl(2)$ case that the θ 'soliton' of the gauge unfixed $sl(2)$ ATM model (see eq. (2.22) of [31]) is half the soliton of the usual SG model.

4. Topological charges, baryons as solitons and confinement

In this section we will examine the vacuum configuration of the cGSG model and the equivalence between the $\mathrm{U}(1)$ spinor current and the topological current (3.22) in the gauge fixed model and verify that the charge associated to the $\mathrm{U}(1)$ current gets confined inside the solitons and kinks of the GSG model obtained in section 2.

It is well known that in $1 + 1$ dimensions the topological current is defined as $J_{\mathrm{top}}^\mu \sim \epsilon^{\mu\nu}\partial_\nu\Phi$, where Φ is some scalar field. Therefore, the topological charge is $Q_{\mathrm{top}} = \int J_{\mathrm{top}}^0 dx \sim \Phi(+\infty) - \Phi(-\infty)$. In order to introduce a topological current we follow the construction adopted in Abelian affine Toda models, so we define the field

$$\theta = \sum_{a=1}^{2} \frac{2\alpha_a}{\alpha_a^2}\theta_a \tag{4.1}$$

where α_a, $a = 1, 2$, are the simple roots of $sl(3,\mathbb{C})$. We then have that $\theta_a = (\theta|\lambda_a)$, where λ_a are the fundamental weights of $sl(3,\mathbb{C})$ defined by the relation [32]

$$2\frac{(\alpha_a|\lambda_b)}{(\alpha_a|\alpha_a)} = \delta_{ab}. \tag{4.2}$$

The exponentials in the fields ϕ_j in the equations (3.1)–(3.11) written as the combinations $(\theta|\alpha_j)$, $j = 1, 2, 3$, where the $\alpha_j's$ are the positive roots of $sl(3,\mathbb{C})$, are invariant under the transformation

$$\theta \rightarrow \theta + 2\pi\mu \quad \text{or} \quad \phi_j \rightarrow \phi_j + 2\pi(\mu|\alpha_j), \tag{4.3}$$

$$\mu \equiv \sum_{n_a \in \mathbb{Z}} n_a \frac{2\vec{\lambda}_a}{(\alpha_a|\alpha_a)}, \tag{4.4}$$

where μ is a weight vector of $sl(3,\mathbb{C})$, these vectors satisfy $(\mu|\alpha_j) \in \mathbb{Z}$ and form an infinite discrete lattice called the weight lattice [32]. However, this weight lattice does not constitute the vacuum configurations of the ATM model , since in the model described by (3.1)–(3.12) for any constants $\theta_a^{(0)}$ and $\eta^{(0)}$

$$\psi_j = \widetilde{\psi}_j = 0, \; \theta_a = \theta_a^{(0)}, \; \eta = \eta^{(0)}, \; \widetilde{\nu} = -\mathbf{m}^2 e^{\eta^{(0)}} x^+ x^- \qquad (4.5)$$

is a vacuum configuration.

We will see that the topological charges of the physical one-soliton solutions of (3.1)–(3.12) which are associated to the new fields φ_a, $a = 1, 2$, of the cGSG model (3.29)–(3.30) lie on a modified lattice which is related to the weight lattice by re-scaling the weight vectors. In fact, the eqs. of motion (3.29)–(3.30) for the field defined by $\varphi \equiv \sum_{a=1}^2 \frac{2\alpha_a}{\alpha_a^2}\varphi_a$, such that $\varphi_a = (\varphi|\lambda_a)$, are invariant under the transformation

$$\varphi \to \varphi + \frac{2\pi}{\beta_0} \sum_{a=1}^2 \frac{q_a}{\nu_a}\frac{2\lambda_a}{(\alpha_a|\alpha_a)}, \; q_a \in \mathbb{Z}. \qquad (4.6)$$

So, the vacuum configuration is formed by an infinite discrete lattice related to the usual weight lattice by the relevant re-scaling of the fundamental weights $\lambda_a \to \frac{1}{\nu_a}\lambda_a$. The vacuum lattice can be given by the points in the plane $\varphi_1 \, \mathrm{x} \, \varphi_2$

$$(\varphi_1 \, , \, \varphi_2) = \frac{2\pi}{3\beta_0}\Big(\frac{2q_1}{\nu_1} + \frac{q_2}{\nu_2} \, , \, \frac{q_1}{\nu_1} + \frac{2q_2}{\nu_2}\Big), \quad q_a \in \mathbb{Z}. \qquad (4.7)$$

In fact, this lattice is related to the one in eq. (2.9) through appropriate parameter identifications. We shall define the topological current and charge, respectively, as

$$J_{\mathrm{top}}^\mu = \frac{\beta_0}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi, \quad Q_{\mathrm{top}} = \int dx J_{\mathrm{top}}^0 = \frac{\beta_0}{2\pi}[\varphi(+\infty) - \varphi(\infty)]. \qquad (4.8)$$

Taking into account the cGSG fields (3.29)–(3.30) and the spinor parameterizations (3.31) the currents equivalence (3.22) of the ATM model takes the form

$$\sum_{j=1}^3 m_j\bar{\psi}^j\gamma^\mu\psi^j \equiv \epsilon^{\mu\nu}\partial_\nu(\zeta_\psi^1\,\varphi_1 + \zeta_\psi^2\,\varphi_2), \qquad (4.9)$$

where $\zeta_\psi^1 \equiv \beta_0^2\nu_1\nu_2(m_\psi^1 d + m_\psi^2 b)$, $\zeta_\psi^2 \equiv \beta_0^2\nu_1\nu_2(m_2 a - m_1 b)$, a, b, d are defined in (3.27)–(3.28), and the spinors are understood to be written in terms of the fields u_j and v_j of (3.31).

Notice that the topological current in (4.9) is the projection of (4.8) onto the vector $\frac{2\pi}{\beta_0}\left(\zeta_\psi^1\lambda_1 + \zeta_\psi^2\lambda_2\right)$.

As mentioned in section 3 the gauge fixing (3.23) preserves the currents conservation laws (3.20). Moreover, the cGSG model was defined for the off critical ATM model obtained after setting $\eta = \mathrm{const.} = 0$. So, for the gauge fixed model it is expected to hold the currents equivalence relation (3.22) written for the spinor parameterizations u_j, v_j and the fields $\varphi_{1,2}$ as is presented in eq. (4.9). Therefore, in order to verify the U(1) current confinement it

is not necessary to find the explicit solutions for the spinor fields. In fact, one has that the current components are given by relevant partial derivatives of the linear combinations of the field solutions, $\varphi_{1,2}$, i.e. $J^0 = \sum_{j=1}^3 m_j \bar{\psi}^j \gamma^0 \psi^j = \partial_x(\zeta_\psi^1 \varphi_1 + \zeta_\psi^2 \varphi_2)$ and $J^1 = \sum_{j=1}^3 m_j \bar{\psi}^j \gamma^1 \psi^j = -\partial_t(\zeta_\psi^1 \varphi_1 + \zeta_\psi^2 \varphi_2)$. In particular the current components J^0, J^1 and their associated scalar field solutions are depicted in figures 3 and 4, respectively, for SG antisoliton and DSG antikink solutions.

It is clear that the charge density related to this U(1) current can only take significant values on those regions where the $x-$derivative of the fields $\varphi_{1,2}$ are non-vanishing. That is one expects to happen with the bag model like confinement mechanism in quantum chromodynamics (QCD). As we have seen the soliton and kink solutions of the GSG theory are localized in space, in the sense that the scalar fields interpolate between the relevant vacua in a limited region of space with a size determined by the soliton masses. The spinor U(1) current gets the contributions from all the three spinor flavors. Moreover, from the equations of motion (3.3)–(3.11) one can obtain nontrivial spinor solutions different from vacuum (4.5) for each set of scalar field solutions (φ_1, φ_2). For example, the solution $\varphi_1 =$soliton, $\varphi_2 = 0$ in section 2.1 implies $\phi_1 = \varphi_1$, $\phi_2 = -\varphi_1$, $\phi_3 = 0$ which substituting into the spinor equations of motion (3.3)–(3.11) will give nontrivial spinor field solutions. Therefore, the ATM model of section 3 can be considered as a multiflavor generalization of the two-dimensional hadron model proposed in [18, 19]. In the last reference a scalar field is coupled to a spinor such that the DSG kink arises as a model for hadron and the quark field is confined inside the bag realizing some properties of the MIT bag model.

In connection to our developments above let us notice that two-dimensional QCD$_2$ has been used as a laboratory for studying the full four-dimensional theory providing an explicit realization of baryons as solitons. It has been conjectured that the low-energy action of QCD$_2$ ($e \gg m_q$, m_q quark mass and e gauge coupling) might be related to massive two dimensional integrable models, thus leading to the exact solution of the strong coupled QCD$_2$ [33]. The baryons in QCD$_2$ may be described as solitons in a bosonized formulation. In the strong-coupling limit the static classical soliton which describes a baryon in QCD$_2$ turns out to be the sine-Gordon soliton. In particular, it has been shown that the $sl(2)$ ATM model describes the low-energy spectrum of QCD$_2$ (1 flavor and N_c colors) and the exact computation of the string tension was performed [34]. A key role has been played by the equivalence between the Noether and topological currents at the quantum level. Moreover, one notice that the SU(n) ATM theory [5, 6] is a $2D$ analogue of the chiral quark soliton model proposed to describe solitons in QCD$_4$ [35], provided that the pseudo-scalars lie in the Abelian subalgebra and certain kinetic terms are supplied for them.

5. Discussion

The generalized sine-Gordon model GSG (2.10)–(2.11) provides a variety of solitons, kinks and bounce type solutions. The appearance of the non-integrable double sine-Gordon model as a sub-model of the GSG model suggests that this model is a non-integrable theory for the arbitrary set of values of the parameter space. However, a subset of values
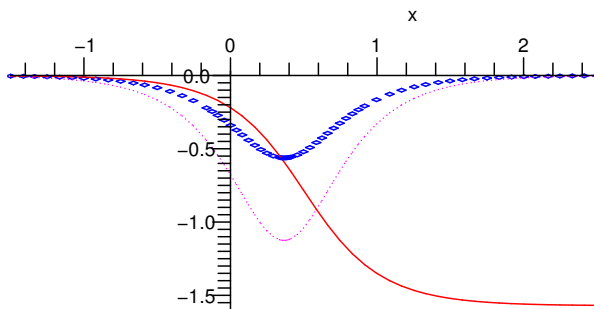
Figure 3: 1-antisoliton and confined current J^μ. The solid curve is the 1-antisoliton ($\frac{\beta_0}{4}\varphi$), the dashdotted curve is J^0 and the curve with losangles is J^1. For $t = 1$, $\mu_1 = \mu_2 = 1$, $d = 1.5$, $v = 0.05$, $\beta_0 = 0.5$, $m_\psi^1 = m_\psi^2 = 1$, $\nu_1 = 1$, $\delta_1 = 1$, $\delta_2 = 2$.
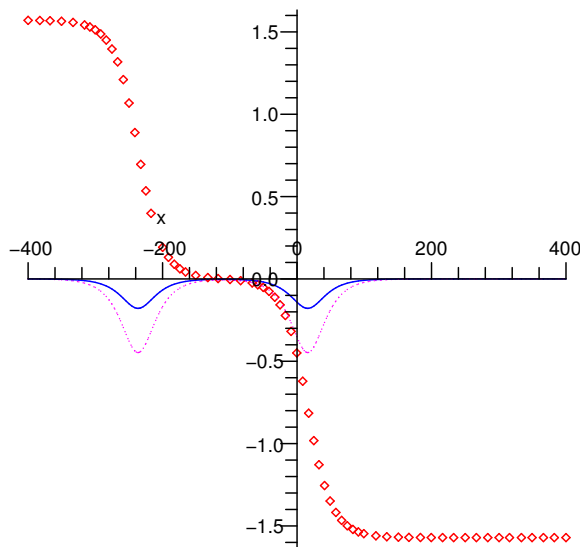
Figure 4: DSG kink solution and confined current J^μ. The curve with losangles is the antikink ($\frac{\beta_0}{4}\varphi$), the dashdotted line is J^0, the solid curve is J^1. For $t = 1$, $\beta_0 = 10^8$, $m_\psi^{1,2} = \mu_1 = -0.0000001$, $\mu_3 := 0.001$, $d = 2$, $\delta_1 = \delta_2 = 1$, $\nu_1 = 1/2$.

in parameter space determine some reduced sub-models which are integrable, e.g. the sine-Gordon submodels of subsections 2.1, 2.2 and 2.3.

In connection to the ATM spinors it was suggested that they are confined inside the GSG solitons and kinks since the gauge fixing procedure does not alter the U(1) and topological currents equivalence (3.22). Then, in order to observe the bag model confinement mechanism it is not necessary to solve for the spinor fields since it naturally arises from the currents equivalence relation. In this way our model presents a bag model like confinement mechanism as is expected in QCD.

The (generalized) massive Thirring model (GMT) is bosonized to the GSG model [7], therefore, in view of the solitons and kinks found above as solutions of the GSG model we expect that the spectrum of the GMT model will contain 4 solitons and their relevant anti-solitons, as well as the kink and antikink excitations. The GMT Lagrangian describes three flavor massive spinors with current-current interactions among themselves. So, the total number of solitons which appear in the bosonized sector suggests that the additional soliton (fermion) is formed due to the interactions between the currents in the GMT sector. However, in subsection 2.3 the soliton masses M_3 and M_4 become the same for the case $\mu_1 = \mu_2$, consequently, for this case we have just three solitons in the GSG spectrum, i.e., the ones with masses M_1, M_2 (subsections 2.1-2.2) and $M_3 = M_4$ (subsection 2.3), which will correspond in this case to each fermion flavor of the GMT model. Moreover, the $sl(3,\mathbb{C})$ GSG model potential (2.6) has the same structure as the effective Lagrangian of the massive Schwinger model with $N_f = 3$ fermions, for a convenient value of the vacuum angle θ. The multiflavor Schwinger model resembles with four-dimensional QCD in many respects (see e.g. [36] and references therein).

The $sl(n,\mathbb{C})$ ATM models may be relevant in the construction of the low-energy effective theories of multiflavor QCD$_2$ with the dynamical fermions in the fundamental and adjoint representations. Notice that in the ATM models the Noether and topological currents and the generalized sine-Gordon/massive Thirring models equivalences take place at the classical [6, 30] and quantum mechanical level [7, 31].

Moreover, the interest in baryons with exotic quantum numbers has recently been stimulated by various reports of baryons composed by four quarks and an antiquark. The existence of these baryons cannot yet be regarded as confirmed, however, reports of their existence have stimulated new investigations about baryon structure (see e.g. [37] and references therein). Recently, there is new strong evidence of an extremely narrow Θ^+ resonance from DIANA collaboration and a very significant new evidence from LEPS. According to Diakonov, "the null results from the new round of CLAS experiments are compatible with what one should expect based on the estimates of production cross sections" [38].

Finally, the spectrum of exotic baryons in QCD$_2$, with SU(N_f) flavor symmetry, has been discussed providing strong support to the chiral-soliton picture for the structure of normal and exotic baryons in four dimensions [39]. The new puzzles in non-perturbative QCD are related to systems with unequal quark masses, so the QCD$_2$ calculation must take into account the SU(N_f)-breaking mass effects, i.e. for $N_f = 3$ it must be $m_s \neq m_{u,d}$. So, in view of our results above, the properties of the GSG and the ATM theories may find some applications in the study of mass splitting of baryons in QCD$_2$ and the understanding of the internal structure of baryons; a work in this direction is under current research [40].

Acknowledgments

A. The zero-curvature formulation of the CATM model

We summarize the zero-curvature formulation of the $sl(3)$ CATM model [5, 6, 22]. Consider the zero curvature condition

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{A.1}$$

The potentials take the form

$$A_+ = -BF^+B^{-1}, \quad A_- = -\partial_- BB^{-1} + F^-, \tag{A.2}$$

with

$$F^+ = F_1^+ + F_2^+, \quad F^- = F_1^- + F_2^-, \tag{A.3}$$

where B and $F_i^\pm$ contain the fields of the model

$$F_1^+ = \sqrt{im_1}\psi_R^1 E_{\alpha_1}^0 + \sqrt{im_2}\psi_R^2 E_{\alpha_2}^0 + \sqrt{im_3}\widetilde{\psi}_R^3 E_{-\alpha_3}^1, \tag{A.4}$$

$$F_2^+ = \sqrt{im_3}\psi_R^3 E_{\alpha_3}^0 + \sqrt{im_1}\widetilde{\psi}_R^1 E_{-\alpha_1}^1 + \sqrt{im_2}\widetilde{\psi}_R^2 E_{-\alpha_2}^1, \tag{A.5}$$

$$F_1^- = \sqrt{im_3}\psi_L^3 E_{\alpha_3}^{-1} + \sqrt{im_1}\widetilde{\psi}_L^1 E_{-\alpha_1}^0 + \sqrt{im_2}\widetilde{\psi}_L^2 E_{-\alpha_2}^0, \tag{A.6}$$

$$F_2^- = \sqrt{im_1}\psi_L^1 E_{\alpha_1}^{-1} + \sqrt{im_2}\psi_L^2 E_{\alpha_2}^{-1} + \sqrt{im_3}\widetilde{\psi}_L^3 E_{-\alpha_3}^0, \tag{A.7}$$

$$B = e^{i\theta_1 H_1^0 + i\theta_2 H_2^0} e^{\widetilde{\nu}C} e^{\eta Q_{ppal}} \equiv b\, e^{\widetilde{\nu}C} e^{\eta Q_{ppal}}. \tag{A.8}$$

$E_{\alpha_i}^n, H_1^n, H_2^n$ and C ($i = 1, 2, 3$; $n = 0, \pm 1$) are some generators of $sl(3)^{(1)}$; Q_{ppal} being the principal gradation operator. The commutation relations for an affine Lie algebra in the Chevalley basis are

$$[H_a^m, H_b^n] = mC\frac{2}{\alpha_a^2}K_{ab}\delta_{m+n,0} \tag{A.9}$$

$$\left[H_a^m, E_{\pm\alpha}^n\right] = \pm K_{\alpha a}E_{\pm\alpha}^{m+n} \tag{A.10}$$

$$\left[E_\alpha^m, E_{-\alpha}^n\right] = \sum_{a=1}^r l_a^\alpha H_a^{m+n} + \frac{2}{\alpha^2}mC\delta_{m+n,0} \tag{A.11}$$

$$\left[E_\alpha^m, E_\beta^n\right] = \varepsilon(\alpha, \beta)E_{\alpha+\beta}^{m+n}; \qquad \text{if } \alpha + \beta \text{ is a root} \tag{A.12}$$

$$[D, E_\alpha^n] = nE_\alpha^n, \qquad [D, H_a^n] = nH_a^n. \tag{A.13}$$

where $K_{\alpha a} = 2\alpha.\alpha_a/\alpha_a^2 = n_b^\alpha K_{ba}$, with n_a^α and l_a^α being the integers in the expansions $\alpha = n_a^\alpha \alpha_a$ and $\alpha/\alpha^2 = l_a^\alpha \alpha_a/\alpha_a^2$, and $\varepsilon(\alpha, \beta)$ the relevant structure constants.

Take $K_{11} = K_{22} = 2$ and $K_{12} = K_{21} = -1$ as the Cartan matrix elements of the simple Lie algebra $sl(3)$. Denoting by α_1 and α_2 the simple roots and the highest one by $\psi(= \alpha_1 + \alpha_2)$, one has $l_a^\psi = 1(a = 1, 2)$, and $K_{\psi 1} = K_{\psi 2} = 1$. Take $\varepsilon(\alpha, \beta) = -\varepsilon(-\alpha, -\beta)$, $\varepsilon_{1,2} \equiv \varepsilon(\alpha_1, \alpha_2) = 1$, $\varepsilon_{-1,3} \equiv \varepsilon(-\alpha_1, \psi) = 1$ and $\varepsilon_{-2,3} \equiv \varepsilon(-\alpha_2, \psi) = -1$.

One has $Q_{ppal} \equiv \sum_{a=1}^2 \mathbf{s}_a \lambda_a^v.H + 3D$, where λ_a^v are the fundamental co-weights of $sl(3)$, and the principal gradation vector is $\mathbf{s} = (1, 1, 1)$ [41].

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